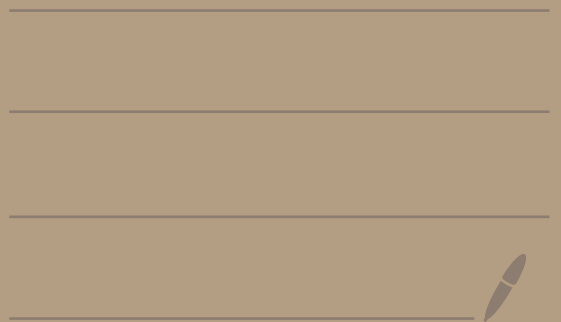


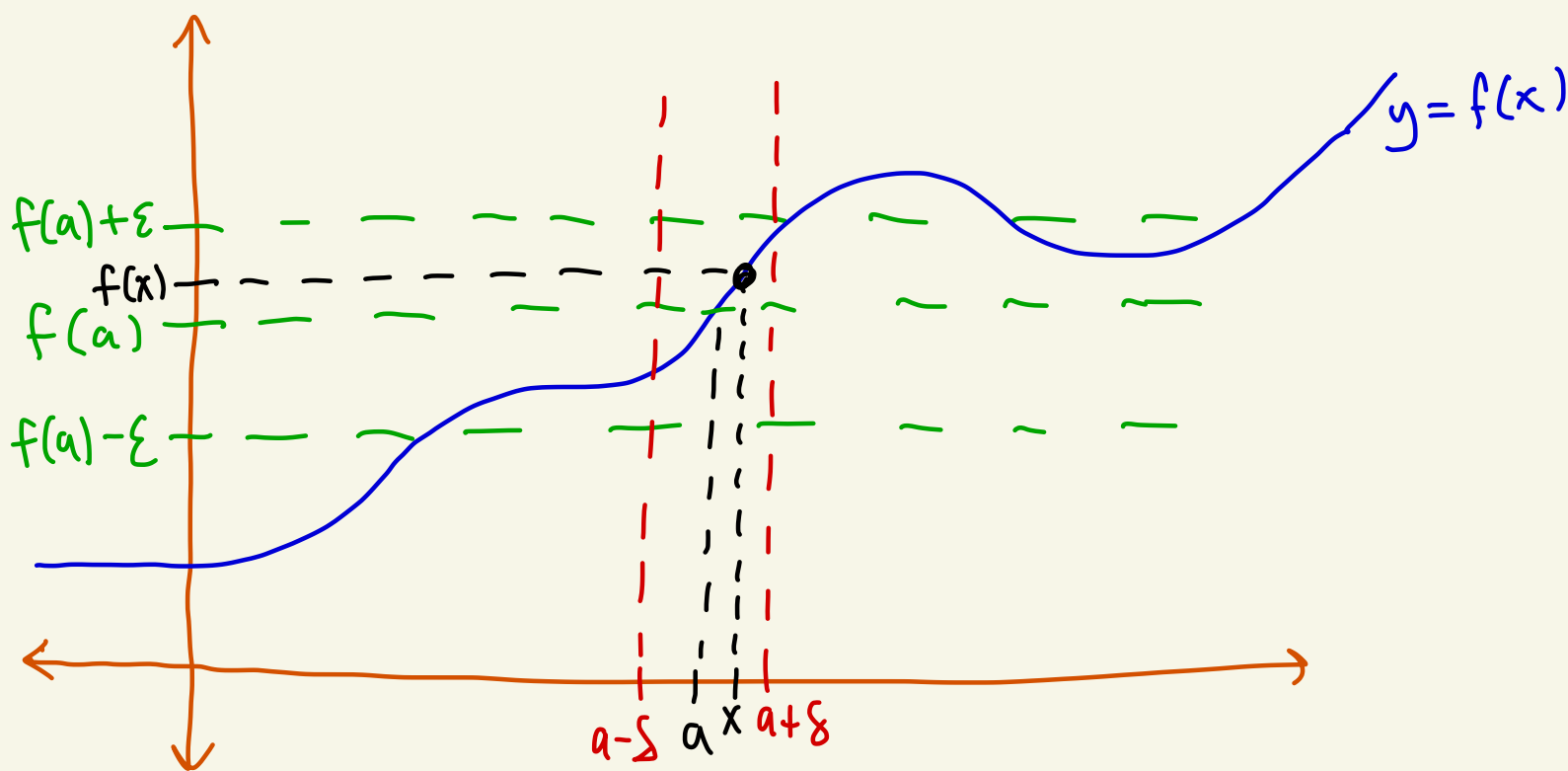
Math 4650

Topic 5 - Continuity



Def: Let $D \subseteq \mathbb{R}$, $a \in D$, and $f: D \rightarrow \mathbb{R}$.

We say that f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x \in D$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.



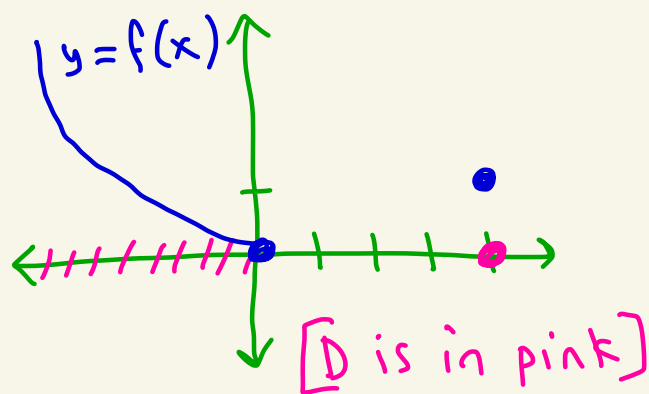
Note: a is in the domain of f . And x is allowed to equal a .

If $A \subseteq D$ and f is continuous at all $a \in A$, then we say that f is continuous on A .

To get an idea of what the above means let's have an example in mind as we go through the two scenarios.

Keep this function in mind for the note below.

$$D = (-\infty, 0] \cup \{4\}$$
$$f: D \rightarrow \mathbb{R}$$
$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 1 & \text{if } x = 4 \end{cases}$$



Note: There are two cases for the definition of continuity of f at a .

Case 1: Suppose that a is a limit point of D .

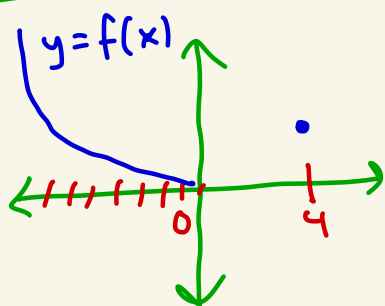
Then we may consider $\lim_{x \rightarrow a} f(x)$.

Then by def, f is continuous at a if and only if

① $\lim_{x \rightarrow a} f(x)$ exists

and ② $\lim_{x \rightarrow a} f(x) = f(a)$

In our example above this is when $-\infty < a \leq 0$



case 2: Suppose a is not a limit point of D .
Then there exists $\delta > 0$ where
 $(a-\delta, a+\delta) \cap D = \{a\}$

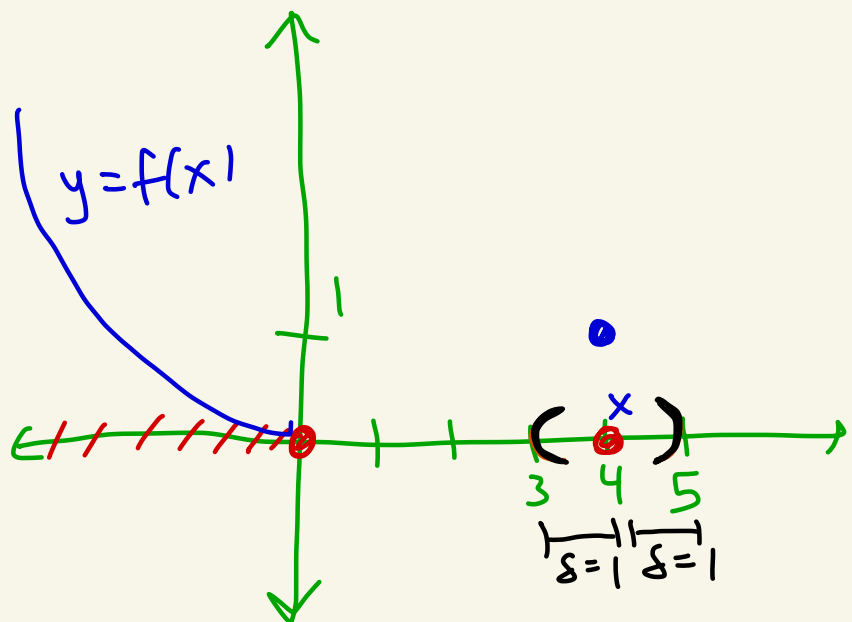
Then, if $x \in D$ and
 $|x-a| < \delta$ we have
that $x=a$ and so
 $|f(x)-f(a)| = |f(a)-f(a)|$
 $= 0 < \varepsilon$

This kind of
point a is
called an
isolated
point of D

for any $\varepsilon > 0$.

So, in this case f is
continuous at $x=a$.

In our example
this happens
when $a=4$.



(D is in red)

Ex: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$.

Let $a \in \mathbb{R}$.

Let's show that f is continuous at a .

We need to show that for any $\varepsilon > 0$
there exists $\delta > 0$ where if $|x - a| < \delta$
then $|x^2 - a^2| < \varepsilon$

Let $\varepsilon > 0$.

Note that

$$|x^2 - a^2| = |x - a| |x + a|$$

we can bound this with δ

we will pick a starting bound on δ to bound this term

Suppose $\delta \leq 1$. ← starting bound

Suppose $|x - a| < \delta \leq 1$.

Then,

$$\begin{aligned} |x + a| &= |x - a + a + a| \\ &= |x - a + 2a| \\ &\leq |x - a| + |2a| \\ &< 1 + 2|a| \end{aligned}$$

This gives

$$|x^2 - a^2| = |x-a||x+a| \\ < |x-a| \cdot (1+2|a|)$$

$$\text{Let } \delta = \min \left\{ \frac{\varepsilon}{1+2|a|}, 1 \right\}.$$

$$\text{Then, } \delta \leq 1 \text{ and } \delta \leq \frac{\varepsilon}{1+2|a|}.$$

Then if $|x-a| < \delta$, we have

$$|x^2 - a^2| < |x-a| \cdot (1+2|a|)$$

$$< \delta \cdot (1+2|a|)$$

$$\leq \frac{\varepsilon}{(1+2|a|)} \cdot (1+2|a|)$$

$$= \varepsilon.$$

since
 $\delta \leq 1$
from
above

So, if $|x-a| < \delta$, then $|x^2 - a^2| < \varepsilon$

Thus, $f(x) = x^2$ is continuous at any $a \in \mathbb{R}$.



Ex: (Dirichlet's function - 1829)

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not continuous at any point in $\mathbb{R}$.

Proof:

Let $a \in \mathbb{R}$ and $\varepsilon = \frac{1}{2}$.

Case 1: Suppose a is rational.

Then, $f(a) = 1$.

Given any $\delta > 0$, there

exists an irrational

number $x \in (a - \delta, a + \delta)$

making $|f(x) - f(a)| = |0 - 1| = 1 > \varepsilon$

So f is not continuous at a .

Case 2: Suppose a is irrational.

Then $f(a) = 0$.

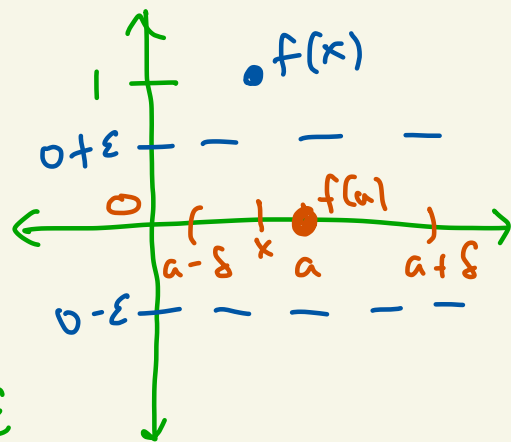
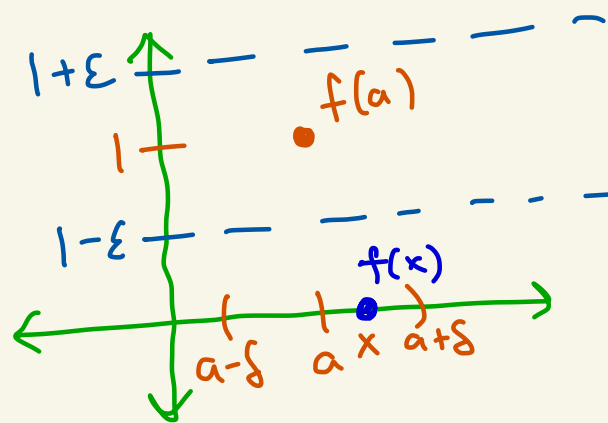
Given any $\delta > 0$, there

exists a rational

number $x \in (a - \delta, a + \delta)$

making $|f(x) - f(a)| = |1 - 0| = 1 > \varepsilon$

So f is not continuous at a .



□

Theorem: Let $D \subseteq \mathbb{R}$ and $a \in D$.

Let $f: D \rightarrow \mathbb{R}$ and $g: D \rightarrow \mathbb{R}$ both be continuous at a . Let $\alpha \in \mathbb{R}$.

Then, αf , $f+g$, $f-g$, and fg are continuous at a .

If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a .

proof:

If a is not a limit point of D , then all the above functions are continuous at a .

Suppose a is a limit point of D .

Then this theorem follows from theorems on limits.

For example, since f and g are continuous at a we know $\lim_{x \rightarrow a} f(x) = f(a)$ and

$$\lim_{x \rightarrow a} g(x) = g(a).$$

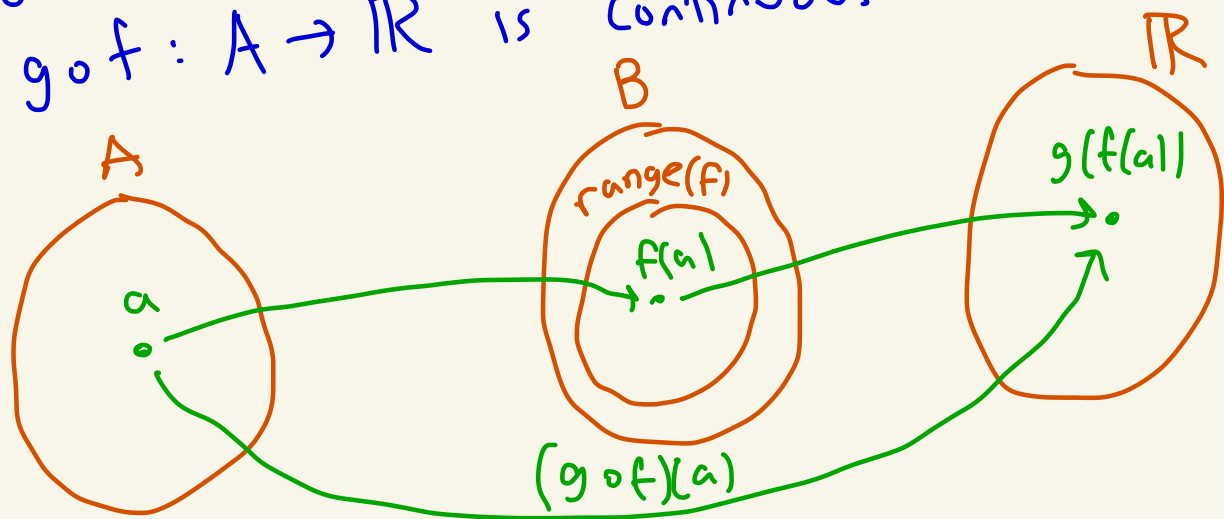
$$\text{Thus, } \lim_{x \rightarrow a} f(x)g(x) = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right] = f(a)g(a)$$

So, fg is continuous at a .

The other proofs are similar.



Theorem: Let $A, B \subseteq \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ be functions such that the range of f is contained in B . If f is continuous at some point $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f: A \rightarrow \mathbb{R}$ is continuous at a .



proof:

Let $\varepsilon > 0$.

Since g is continuous at $f(a)$ there exists $\delta_1 > 0$ where if $y \in B$ and $|y - f(a)| < \delta_1$, then $|g(y) - g(f(a))| < \varepsilon$.

Since f is continuous at a there exists $\delta > 0$ where if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \delta_1$.

Since the range of f is contained in B ,

we have that if $x \in A$ and $|x - a| < \delta$, then $f(x) \in B$ and $|f(x) - f(a)| < \delta$, which will give $|g(f(x)) - g(f(a))| < \varepsilon$.

Thus, $g \circ f$ is continuous at a .



Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \rightarrow \mathbb{R}$ and $a \in D$. Then f is continuous at a if and only if $\lim_{n \rightarrow \infty} f(x_n) = f(a)$ for every sequence (x_n) contained in D with $x_n \rightarrow a$.

proof: HW



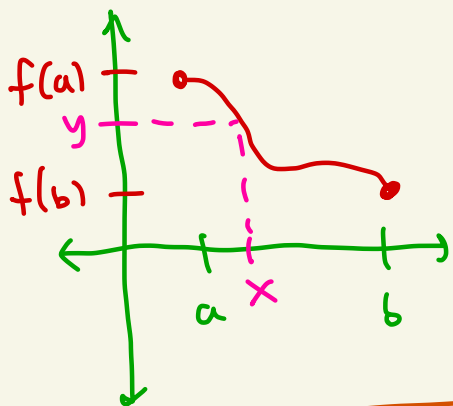
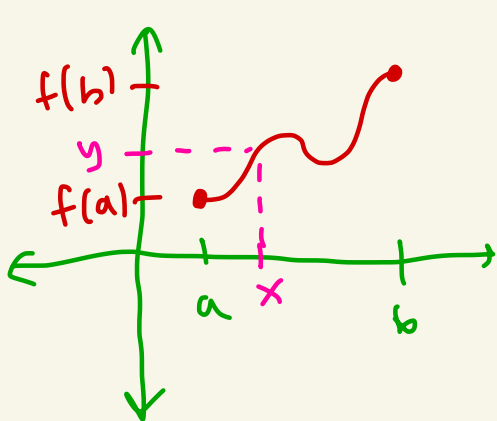
Theorem (Intermediate value theorem)

Let f be continuous on $[a, b]$ where $a < b$.

If for some $y \in \mathbb{R}$ we have either

$$f(a) < y < f(b) \quad \text{or} \quad f(b) < y < f(a)$$

then there exists x with $a < x < b$ and $f(x) = y$.



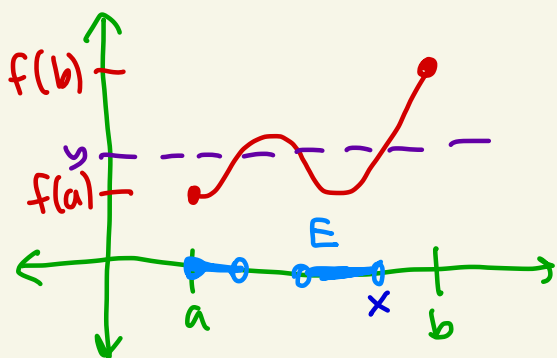
proof:

We will prove it when $f(a) < y < f(b)$.

The other case is similar.

Define

$$E = \{t \mid a \leq t \leq b \text{ and } f(t) < y\}$$



Note that $a \in E$ since $a \leq a \leq b$ and $f(a) < y$.

So, E is not empty.

Since E is bounded above by b we know $x = \sup(E)$ exists

From HW 1 we know that if $A \subseteq B$ then
 $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.

Since $E \subseteq [a, b]$ we know
 $a \leq \inf(E) \leq \sup(E) \leq b$

So, $a \leq x \leq b$.

Next we show that $a < x < b$.

We know that $y < f(b)$.

Since f is continuous at b , there
exists $\delta > 0$ so that if

$x \in [a, b]$ and $|x - b| < \delta$,

then $|f(x) - f(b)| < \underbrace{f(b) - y}_{\substack{\text{positive} \\ \text{since } y < f(b)}}$

So, if $b - \delta < x \leq b$, then

$$-(f(b) - y) < f(x) - f(b) < f(b) - y$$

That is if $b - \delta < x \leq b$, then

$$y < f(x) < 2f(b) - y$$

So, if $b - \delta < x \leq b$, then $y < f(x)$.

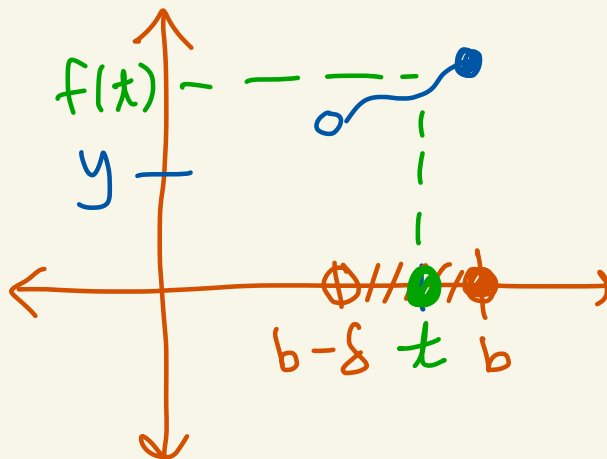
So,

$$(b-\delta, b] \cap E = \emptyset$$

Thus,

$$x \leq b-\delta < b.$$

$$\text{So, } x < b.$$



A similar computation gives $a < x$.

We know $f(a) < y$

Since f is continuous at a there exists $\delta > 0$ where if $a \leq t < a+\delta$, then

$$|f(t) - f(a)| < y - f(a).$$

That is, for $a \leq t < a+\delta$,

$$-(y - f(a)) < f(t) - f(a) < y - f(a)$$

or

$$-y + 2f(a) < f(t) < y$$

$$\text{So, } [a, a+\delta) \subseteq E,$$

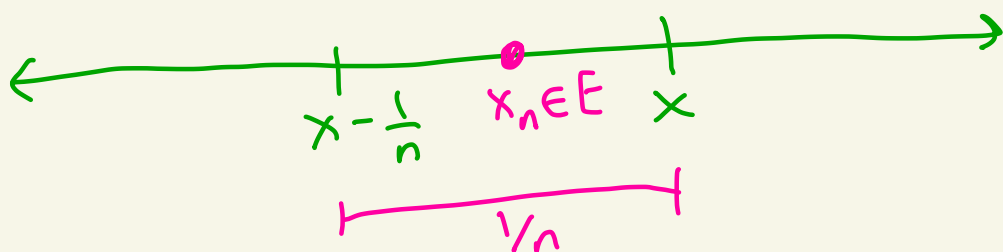
Thus since $x = \sup(E)$ we must have $a < x$.

Thus, $a < x < b$.

Next we will show that $f(x) = y$.

First we show that $f(x) \leq y$.

By the inf-sup theorem, since $x = \sup(E)$,
for each $n \in \mathbb{N}$ there exists $x_n \in E$
such that $x - \frac{1}{n} < x_n \leq x$.



This gives us a sequence (x_n) contained in E
that converges to x .

For every n , since $x_n \in E$ we know $f(x_n) < y$.

Since f is continuous at x , and x is
a limit point of $E \subseteq [a, b]$, we know

$$\lim_{n \rightarrow \infty} f(x_n) = f(x).$$

By Hw 2, since $f(x_n) < y$ for all n

and $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ we

know that $f(x) = \lim_{n \rightarrow \infty} f(x_n) \leq y$.

$$\text{So, } f(x) \leq y.$$

Now we rule out the case $f(x) < y$.

Suppose $f(x) < y$,

Then since f is continuous at x there exists $\delta > 0$ where if $t \in [a, b]$ and $|t - x| < \delta$, then

$$|f(t) - f(x)| < \underbrace{y - f(x)}_{\substack{\text{positive} \\ \text{since } f(x) < y}}$$

In the above we can assume that $\delta < b - x$ by shrinking it if needed.

Thus, if $x - \delta < t < x + \delta < b$

$$\text{then } -(y - f(x)) < f(t) - f(x) < y - f(x)$$

So if $x - \delta < t < x + \delta$,

$$\text{then } -y + 2f(x) < f(t) < y$$

But then

$$(x-\delta, x+\delta) \subseteq E$$

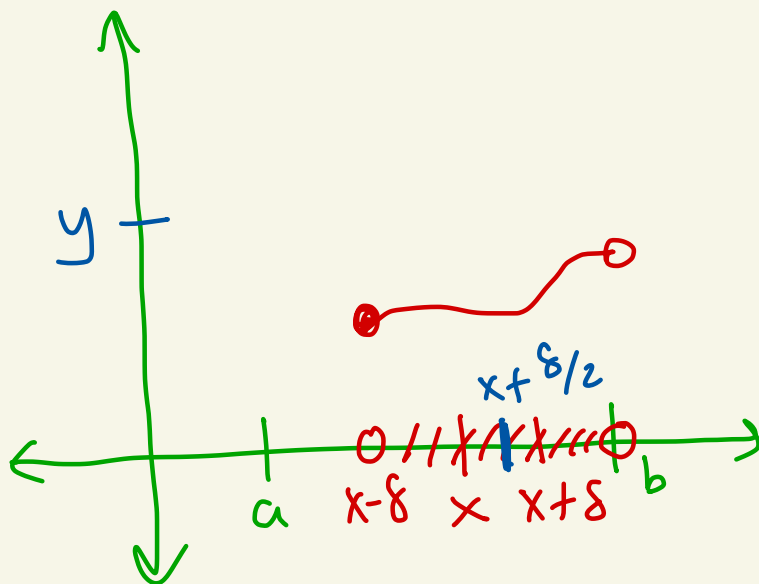
So for example

$$x + \frac{\delta}{2} \in E$$

This contradicts
the fact that $x = \sup(E)$.

Hence $f(x) < y$ is impossible

Therefore $f(x) = y$.

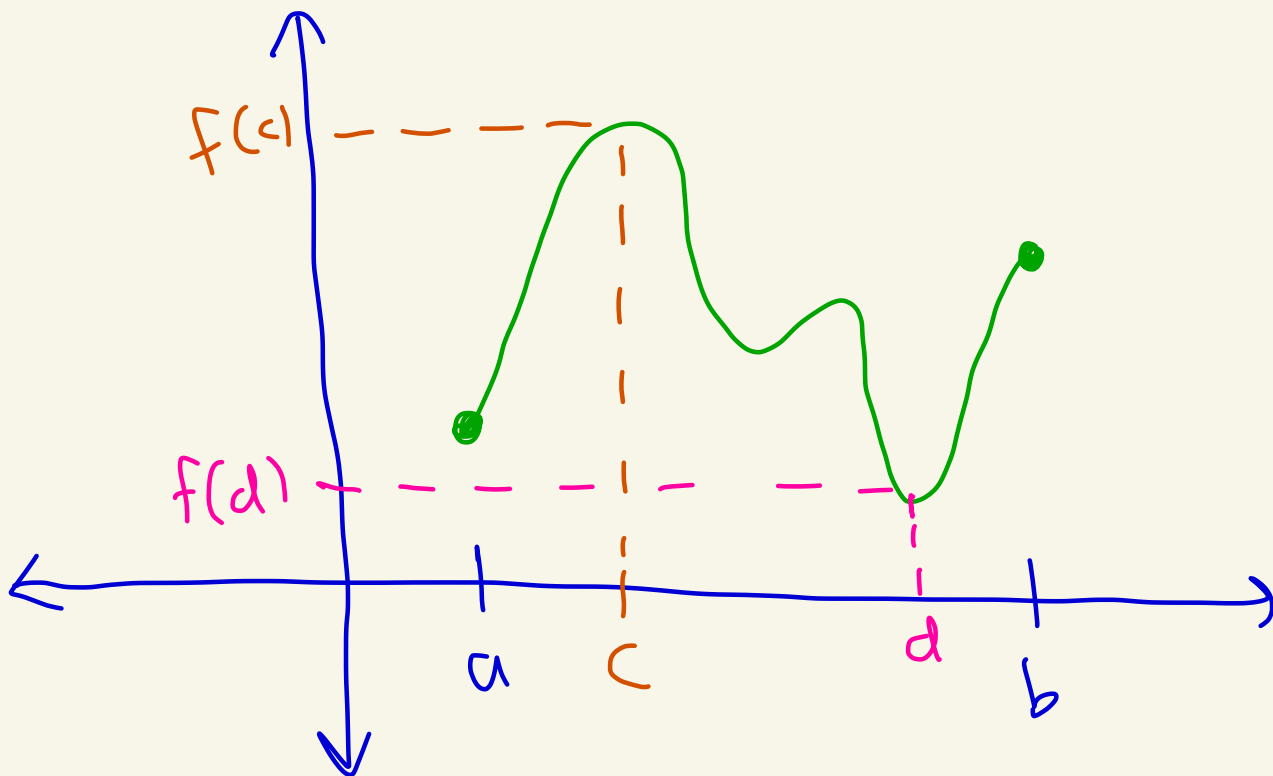


Theorem: Let f be continuous on $[a, b]$ with $a < b$.

Then f attains its maximum and minimum values on $[a, b]$.

That is, there exists $c \in [a, b]$ where $f(c) \geq f(x)$ for all $a \leq x \leq b$

And there exists $d \in [a, b]$ where $f(d) \leq f(x)$ for all $a \leq x \leq b$.



proof:

We will prove that f attains its maximum. For the minimum, repeat the proof with $-f$ in place of f .

Consider

$$S = \{f(x) \mid a \leq x \leq b\}$$

Let's show that S is bounded from above.

Suppose that it is not.

Then for each $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ with $f(x_n) > n$.

Since $a \leq x_n \leq b$ for each n we get that (x_n) is a bounded sequence.

By Bolzano-Weierstrass there is a convergent subsequence

$$x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \dots$$

with $n_1 < n_2 < n_3 < n_4 < \dots$

Suppose $\lim_{n_k \rightarrow \infty} x_{n_k} = C$.

Since $a \leq x_{n_k} \leq b$ for all n_k ,
by HW 2 we get $a \leq C \leq b$.

Since f is continuous at C
we get $f(C) = \lim_{x \rightarrow C} f(x)$

$$= \lim_{n_k \rightarrow \infty} f(x_{n_k})$$

HW 4
it's essentially
the function-sequence
limit theorem

But this is a contradiction because
 $f(x_{n_k}) \geq n_k$ for all k

and $n_k \rightarrow \infty$.

So, $\lim_{n_k \rightarrow \infty} f(x_{n_k})$ does not exist.

Therefore we must have that S is bounded from above.

So, $M = \sup(S)$ exists.

By the inf-sup theorem,
for each $m \in \mathbb{N}$ there exists y_m with $a \leq y_m \leq b$ and $M - \frac{1}{m} < f(y_m) \leq M$.

Thus, $\lim_{m \rightarrow \infty} f(y_m) = M$. $\leftarrow$

Since (y_m) is a bounded sequence there exists a convergent subsequence (y_{m_k}) .

Suppose $y_{m_k} \rightarrow d$.

Since $a \leq y_{m_k} \leq b$ for all m_k we know by HW 2 that $a \leq d \leq b$.

Thus, by the continuity of f

Let $\varepsilon > 0$.
Pick N s.t.
 $\frac{1}{N} < \varepsilon$. Then
if $m \geq N$, then
 $|f(y_m) - M| < \frac{1}{m}$
 $\leq \frac{1}{N} < \varepsilon$

$$f(d) = \lim_{m_l \rightarrow \infty} f(y_{m_l}) = M.$$

↑
HW 4

HW 2
since $f(y_m) \rightarrow M$
we know $f(y_{m_l}) \rightarrow M$

Since $f(d) = M = \sup(S)$ we know
 $f(d) \geq f(x)$ for all $a \leq x \leq b$.

So f attains its maximum
on $[a, b]$ at d .

